General Relativity

Final Exam
1/11/2013
Please write your first and last name and your student number on the first page.

Problem #1: Rindler space

Consider the 2-dimensional metric

$$ds^2 = x^2 dt^2 - dx^2$$

defined for $x > 0$ and for all t.

1. Compute the Christoffel symbols.
2. How many independent components does the Riemann tensor have in 2 dimensions? Compute the Riemann tensor of this metric.
3. Consider a particle sitting at $x(t) = x_0 > 0$ for all t. Write down the velocity vector u^μ of the particle in these coordinates, and compute the relativistic acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$. Using these results, show that the particle experiences a force towards $x = 0$ and compute this force in terms of x_0. Check that the force blows up if x_0 is taken to be close to $x = 0$.
4. Now consider a massive particle moving *inertially* (i.e. it is in free fall) in this spacetime. Derive the equations of motions for the particle.
5. Let us say that at $t = 0$ the particle is released with zero initial velocity from a point $x = x_0 > 0$ and then it moves inertially. Using the equations you derived in the previous item, find the orbit of the particle by writing $x(\tau), t(\tau)$, where τ is the proper time.
6. Show that in *finite* amount of proper time the particle reaches $x = 0$, compute this amount of proper time as a function of x_0.
7. Using the previous result, write the orbit of the particle in terms of *coordinate time* t, i.e. give $x(t)$. Check that as a function of t, the particle never reaches the line $x = 0$, but rather it asymptotically approaches it (this is in contrast to what we found in item 6., in terms of *proper time*).
8. Now we will see that the spacetime can be extended smoothly past the "horizon" $x = 0$. Find the orbits of left- and right-moving light rays, and show that they can be written as $u = xe^{-t} = \text{const}$ and $v = -xe^t = \text{constant}$ respectively. Show that in these coordinates the metric takes the form

   $$ds^2 = du dv$$

   Notice that from the definition of u, v, and since the original spacetime is constrained to $x > 0$, we have that $u > 0, v < 0$. However, nothing prevents us from extending the metric to all values of u, v.
9. Finally define $u = T + X, v = T - X$ and show that the metric takes the form

   $$ds^2 = dT^2 - dX^2$$

   check that the "horizon" $x = 0$ that we found in the (t, x) coordinates, corresponds to the ligthcone $T - X$ (for $T > 0$)

Problem #2: Einstein Static Universe

Consider a closed homogeneous and isotropic Universe (i.e. described by the Robertson-Walker metric with $k = +1$)

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

which is filled with pressureless matter of density ρ and cosmological constant Λ.

1. Find what is the relation between ρ and Λ in order for the scale factor to be time-independent. This leads to a *static* model of the Universe (no expansion).
2. Show that this model is unstable under perturbations (i.e. small perturbations of the density will cause $a(t)$ to increase/decrease far from the original equilibrium value).

Problem #3: Motion in the background of a charged black hole

Consider the metric

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which describes the spacetime around a black hole of mass M and electric charge Q.

1. Write down the equations of motion for a massive particle in free fall in this geometry.
2. Write the effective potential for the radial motion of the particle.
3. Using these equations, check that a freely falling particle will never reach $r = 0$ (not even according to its proper time).
4. A particle is released from infinity with zero velocity and moves *radially* inwards (i.e. it has zero angular momentum). Find the smallest value of r that the particle will reach.